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# Topology and $\theta$ dependence in finite temperature $G_2$ lattice gauge theory

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**ABSTRACT:** In this work we study the topological properties of the  $G_2$  lattice gauge theory by means of Monte Carlo simulations. We focus on the behaviour of topological quantities across the deconfinement transition and investigate observables related to the  $\theta$  dependence of the free energy. As in  $SU(N)$  gauge theories, an abrupt change happens at deconfinement and an instanton gas behaviour rapidly sets in for  $T > T_c$ .

**KEYWORDS:** Lattice Gauge Field Theories, Nonperturbative Effects, Confinement.

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## 1 Introduction

A large part of the standard particle phenomenology depends on phenomena like color confinement and chiral symmetry breaking, however a complete understanding of these nonperturbative phenomena is still lacking. In order to better figure out these phenomena, a common strategy has been to extend the theory under study beyond its natural setting, in order to find regularities or more general patterns underlying the usual structure.

As an example, real life Quantum Chromodynamics (QCD) is a theory with gauge group  $SU(3)$  and fermions with particular mass values, however, in order to grasp some intuition on its strongly coupled dynamics, it is convenient to look at theories with fermions of arbitrary masses, with particular attention to the extremal cases of massless fermions and pure  $SU(3)$  theory. In the same way, it is convenient to think of the  $SU(3)$  theory just as a particular realization of the general theory with gauge group  $SU(N)$ , which is, e.g., the natural setting for techniques like the large  $N$  expansion. Although these approaches were not capable of providing quantitatively reliable predictions, their qualitative indications are of the utmost importance to get some understanding of the QCD nonperturbative physics.

An even more drastic extension is obtained by considering gauge theories with an arbitrary gauge group. In this paper we will study some properties of the theory with the exceptional group  $G_2$  as gauge group. In order to motivate this apparently bizarre choice we need some background.

In pure gauge theory, at temperature  $T$ , the free energy  $F_{Q\bar{Q}}(\vec{r})$  of a couple of static color-anticolor charges at distance  $\vec{r}$  is given by the expression [1]

$$\exp(-F_{Q\bar{Q}}(\vec{r})/T) = \langle P(0)P(\vec{r})^* \rangle, \quad (1.1)$$

where  $P(\vec{r})$  is the Polyakov loop. In the Lattice Gauge Theory (LGT, [2]) setting this can be written as

$$P(\vec{r}) = \text{Tr} \prod_{t=0}^{N_t-1} U_0(\vec{r}, t), \quad (1.2)$$

the  $U_\mu(\vec{r}, t)$ s being the elementary parallel transports along the links of the lattice and  $N_t$  the number of lattice elements in the compactified temporal direction. From Eq. (1.1) it follows that a couple of static color charges can be separated at arbitrary distances only if the thermal average of the Polyakov loop is different from zero:  $\langle P \rangle \neq 0$ . Let us now consider for every  $\vec{r}$  the following transformation

$$U_0(\vec{r}, \bar{t}) \rightarrow U'_0(\vec{r}, \bar{t}) = Z U_0(\vec{r}, \bar{t}) , \quad (1.3)$$

where  $\bar{t}$  is a fixed time slice of the lattice, and  $Z$  is an element of the center of the gauge group (independent of  $\vec{r}$ ). It is simple to show that the Wilson lattice action (see Sec. 3) is invariant under the transformation Eq. (1.3), while clearly the average of the Polyakov loop transforms as  $\langle P \rangle \rightarrow Z \langle P \rangle$ . As a consequence deconfinement can be associated to the spontaneous breaking of the symmetry Eq. (1.3), i.e. of the center symmetry (see e.g. [3] for much more details).

Although this picture of confinement appears quite satisfying and leads to nontrivial predictions (like e.g. the Svetitsky-Yaffe conjecture [4] on the universality class of the deconfinement transition), it clearly does not cover two class of theories:

1. theories whose action is not invariant under Eq. (1.3), like theories with fermions in the fundamental color representations;
2. theories with a trivial gauge group center.

Theories with fermions are notoriously computationally demanding, so they are not the first choice for a study that, depending on the observable to be monitored, could require high statistics. Concerning the theories of the second class, the simplest group with trivial center that comes to mind is  $SO(3)$ , however in this case (and more generally for all the  $SO(N)$  groups) large lattice artefacts make a systematic lattice investigation problematic, see e.g. [5]. A particularly interesting alternative proved to be the gauge group  $G_2$ . Beyond having trivial center,  $G_2$  presents two peculiar features:

- a) it is simply connected;
- b) a charge in the fundamental representation (**7**) can be screened by charges in the adjoint representation (**14**).

The first of these properties is interesting when studying confinement, since it means that a  $G_2$  gauge theory does not support topologically stable vortex configurations, thus at least one of the possible models of color confinement (i.e. the vortex picture, see e.g. [3]) requires non-trivial modifications to be applied in the  $G_2$  setting. For this reason, together with the fact that  $G_2$  has the low rank value 2, the  $G_2$  gauge theory was often used as a testbed for possible confinement mechanisms (see e.g. [6–10]). The second of the aforementioned properties<sup>1</sup> makes the  $G_2$  pure gauge theory quite similar to a theory coupled to matter, in which color is confined but the area law for the Wilson loop is

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<sup>1</sup>which follows from the fact that the Clebsch-Gordan series of the product  $\mathbf{7} \otimes \mathbf{14} \otimes \mathbf{14} \otimes \mathbf{14}$  contains a singlet, see e.g. [11].

not valid (or, equivalently, the asymptotic string tension vanishes) because of the string breaking phenomenon.

Bearing all this in mind, it is not surprising that the  $G_2$  lattice gauge theory was actively investigated in the past, both at zero and finite temperature [11–22]. What is probably surprising is that the results of these analysis gave a picture much similar to that of standard  $SU(N)$  theory: the spectrum of the  $G_2$  theory at zero temperature is composed only of color neutral objects [11, 12, 15], the string tension at intermediate distances (i.e. before string breaking [20]) satisfies Casimir scaling [13, 18, 20], a first order deconfinement transition is present [14, 16, 22], (quenched) chiral symmetry is broken in the low temperature phase and restored above the critical temperature [19], the topological susceptibility is suppressed above deconfinement [21], propagators [17] and thermodynamical observables (like e.g. pressure and trace anomaly) [22] do not show any qualitative difference with respect to the  $SU(N)$  case.

In this paper we will study the topological properties of the  $G_2$  LGT near the deconfinement transition, with particular attention to observables related to the functional dependence of the free energy on the  $\theta$  angle. As will be recalled in more detail in the next section, in  $SU(N)$  one expects an abrupt change of the functional form of the free energy at deconfinement, switching from the large  $N$  behaviour in the low- $T$  phase to an instanton gas behaviour in the high- $T$  phase. It is *a priori* clear that such an argument, based on the large  $N$  analysis of  $SU(N)$  gauge theories cannot be directly applied to the case of the  $G_2$  LGT, nevertheless our results indicate that also in this case the  $G_2$  theory resembles very much the  $SU(N)$  case.

## 2 Topology and $\theta$ dependence in $SU(N)$ gauge theories

In this section we will summarize, for the convenience of the reader, some basics facts about topology and  $\theta$  dependence in  $SU(N)$  theories, in such a way to make the comparison with the  $G_2$  case simpler.

The euclidean Lagrangian density of the  $SU(N)$  (continuum) gauge theory is

$$\mathcal{L}_\theta = \frac{1}{2} \text{Tr}[F_{\mu\nu} F_{\mu\nu}] - i\theta q(x) , \quad q(x) = \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}[F_{\mu\nu}(x) F_{\rho\sigma}(x)] \quad (2.1)$$

and the associated free energy density can be computed by means of the relation

$$F(\theta, T) = -\frac{1}{V_4} \log \int [\mathcal{D}A] \exp \left( - \int_0^{1/T} dt \int_V d^3x \mathcal{L}_\theta \right) , \quad V_4 = T/V , \quad (2.2)$$

where  $T$  is the temperature and  $V$  the spatial volume. The topological charge  $Q = \int q(x) d^4x$  is odd under parity transformation, thus at  $\theta = 0$  we have  $\langle Q^{2n+1} \rangle_{\theta=0} = 0$ , since by the Vafa-Witten theorem parity cannot be spontaneously broken [23]. As a consequence, assuming analyticity in  $\theta = 0$ ,  $F(\theta, T)$  is an even function of  $\theta$  and can be expanded in the following form [24]

$$F(\theta, T) - F(0, T) = \frac{1}{2} \chi(T) \theta^2 [1 + b_2(T) \theta^2 + b_4(T) \theta^4 + \dots] . \quad (2.3)$$

The topological susceptibility  $\chi(T)$  and the coefficients  $b_{2n}$  can be computed by using the momenta of the topological charge distribution at  $\theta = 0$  as

$$\begin{aligned}\chi(T) &= \frac{\langle Q^2 \rangle_{\theta=0}}{V_4} & b_2 &= -\frac{\langle Q^4 \rangle_{\theta=0} - 3\langle Q^2 \rangle_{\theta=0}^2}{12\langle Q^2 \rangle_{\theta=0}} \\ b_4 &= \frac{\langle Q^6 \rangle_{\theta=0} - 15\langle Q^2 \rangle_{\theta=0}\langle Q^4 \rangle_{\theta=0} + 30\langle Q^2 \rangle_{\theta=0}^3}{360\langle Q^2 \rangle_{\theta=0}}.\end{aligned}\tag{2.4}$$

At finite temperature, instanton calculus does not suffer from infrared divergences and can be used to gain some insight into the functional form of  $F(\theta, T)$ . The idea of the dilute instanton gas approximation is to replace the path-integral expression of the partition function by the sum over an ensemble of noninteracting instantons and anti-instantons. Denoting by  $D^{-1/4}$  the typical size of an instanton, we get [25]

$$\begin{aligned}&\int [\mathcal{D}A] \exp \left( - \int_0^{1/T} dt \int_V d^3x \mathcal{L}_\theta \right) \approx \\ &\approx \sum_{n_+, n_- = 0}^{\infty} \frac{1}{n_+! n_-!} (V_4 D)^{n_+ + n_-} \exp \left( - \frac{8\pi^2}{g^2} (n_+ + n_-) + i\theta(n_+ - n_-) \right) = \\ &= \exp \left[ 2(V_4 D) e^{-8\pi^2/g^2} \cos \theta \right],\end{aligned}\tag{2.5}$$

and thus (using the one loop running coupling constant and  $D \sim T^4$ )

$$F(\theta, T) - F(0, T) = \chi(T)(1 - \cos \theta)\tag{2.6}$$

$$\chi(T) \sim T^4 \exp \left[ - 8\pi^2/g^2(T) \right] \sim T^{-\frac{11}{3}N+4}\tag{2.7}$$

$$b_2 = -\frac{1}{12} \quad b_4 = \frac{1}{360} \quad b_{2n} = (-1)^n \frac{1}{(2n+2)!}.\tag{2.8}$$

The approximation in Eq. (2.5) is expected to be reliable at high temperatures; in this regime the instanton gas predicts a strong suppression of the topological susceptibility, which gets stronger when increasing the number of colors. This is in fact what is observed in numerical simulations [26–31]: the topological susceptibility stays constant for  $T \lesssim T_c$ , while it drops toward zero at deconfinement, in qualitative accordance with Eq. (2.7).

It has to be stressed that Eq. (2.7) involves two different approximations: the instanton gas approximation and the perturbative one, thus it cannot be expected to be valid in the strongly coupled region near deconfinement. Only the instanton gas approximation is instead used to obtain the  $b_{2n}$  values in Eq. (2.8).

Another approach that gives us some information on  $F(\theta, T)$  is the 't Hooft large  $N$  limit [32, 33], which is expected to be reliable at low temperature. If we do not want the  $\theta$  dependence to be washed out by the  $N \rightarrow \infty$  limit, we have to impose that the two terms in the Lagrangian Eq. (2.1) scale in the same way with  $N$ . Remembering that  $g^2 = \mathcal{O}(1/N)$ , we obtain the large  $N$  scaling form of the free energy [34, 35]:

$$F(\theta, T) = N^2 \mathcal{F}(\bar{\theta}, T), \quad \bar{\theta} = \theta/N.\tag{2.9}$$

By comparing the power series in  $\theta$  of the left and right hand side we get

$$\chi(T) = \bar{\chi}(T) + \mathcal{O}(1/N^2) \quad b_{2n}(T) = \bar{b}_{2n}(T)/N^{2n} , \quad (2.10)$$

where  $\bar{\chi}$  and  $\bar{b}_{2n}$  are the coefficient of the expansion of  $\mathcal{F}$  in power of  $\bar{\theta}$  analogous to Eq. (2.3). Several lattice measurements of  $\chi(T)$  and  $b_2$  exist at  $T = 0$  [36–39], that nicely follow the large  $N$  scaling Eq. (2.10).

The functional dependence of the free energy on  $\theta$  is expected to be completely different in the low and high temperature phases: at low temperature the natural variable is  $\bar{\theta} = \theta/N$  and this, together with the  $2\pi$  periodicity in  $\theta$ , suggests  $F$  to be a multibranched function of the form [34, 35]

$$F(\theta, T) = N^2 \min_k H \left( \frac{\theta + 2\pi k}{N}, T \right) . \quad (2.11)$$

On the other hand, instanton gas predicts the form Eq. (2.6) of the free energy, which is thus expected to be analytic in  $\theta$  in the high temperature phase.

Although large  $N$  methods and instanton calculus are reliable for  $T \ll T_c$  and  $T \gg T_c$  respectively, it appears natural to guess that the change of regime happens exactly at deconfinement. In order to clarify this issue, in [40] the behaviour of  $b_2$  across the deconfinement transition was numerically investigated for  $SU(3)$  and  $SU(6)$  LGTs. The advantage of  $b_2$  with respect to the topological susceptibility  $\chi(T)$  is that we have a clear cut distinction between the two possible behaviours:  $b_2$  scales as  $1/N^2$  or is independent of  $N$ . Moreover, as previously noticed, the value of  $b_2$  in the high temperature regime is unambiguously predicted by the instanton gas approximation, all the uncertainties related to perturbation theory being factorized into  $\chi(T)$ . The value of  $b_2$  observed at  $T \approx 0.95T_c$  is compatible with the one at  $T = 0$  and scales according to Eq. (2.10); above the transition ( $T \gtrsim 1.05T_c$ ) the value of  $b_2$  does not scale with  $N$ , thus indicating that the relevant variable is not  $\bar{\theta}$  but just  $\theta$ . The instanton gas prediction for  $b_2$  and  $b_4$  turned out to be well satisfied for temperature just slightly above deconfinement ( $T \gtrsim 1.1T_c$ ). These properties are also reproduced by model calculations in QCD-like theories [9, 41–43].

### 3 $G_2$ lattice gauge theory

In LGT [2] the elementary objects are the parallel transports along the links of the lattice, that will be denoted by  $U_\mu(x)$  and are elements of the gauge group. The group  $G_2$  can be identified with the group of the automorphism of the octonions [44], which is isomorphic to the subgroup of  $SO(7)$  that leaves invariant a specific 3–form [45]: a  $7 \times 7$  real matrix  $M$  is an element of  $G_2$  if and only if  $M \in SO(7)$  and

$$T_{abc} = T_{a'b'c'} M_{aa'} M_{bb'} M_{cc'} , \quad (3.1)$$

where  $T_{abc}$  is the completely antisymmetric tensor whose non-vanishing elements (up to permutations) are given by<sup>2</sup> [16]

$$T_{123} = T_{176} = T_{145} = T_{257} = T_{246} = T_{347} = T_{365} = 1. \quad (3.2)$$

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<sup>2</sup>The explicit form of the  $T$  tensor is base dependent, for other possible choices see, e.g. [13].

In our simulations we adopted the standard Wilson plaquette action [2]

$$S_W = -\beta \sum_x \sum_{0 \leq \mu, \nu \leq 3} \text{Tr} P_{\mu\nu}(x) , \quad (3.3)$$

where  $P_{\mu\nu}$  is the product of four links around an elementary plaquette:

$$P_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x) . \quad (3.4)$$

Notice that in the literature two different conventions on the value of  $\beta$  exist for the  $G_2$  theory: the one in Eq. (3.3) and the one that corresponds to the change  $\beta \rightarrow \beta/7$  in Eq. (3.3).

A particularly convenient basis for the  $G_2$  algebra was constructed in [46], which can be adopted to easily identify the  $SU(2)$  subgroups to be used in a Monte-Carlo update *a la* Cabibbo-Marinari [47]. The application of this method to the  $G_2$  theory is not, however, completely trivial: as can be explicitly seen by looking at the expressions in App. A of [13], only three  $SU(2)$  subgroups are embedded in  $G_2$  in a way simple enough to be efficiently used in a standard heatbath/overrelaxation update [48–50]. These  $SU(2)$  subgroups do not cover completely the  $G_2$  group and, to ensure the ergodicity of the update algorithm, random gauge transformations has to be also applied.

Before going on, a small digression is necessary on the normalization of the topological charge. Instantons and topological charge are usually discussed in the setting of  $SU(N)$  gauge theories, in which the topological charge density is given by Eq. (2.1) and the normalization is fixed by the requirement that the topological charge  $Q = \int q(x) dx$  has to be equal to the winding number associated to  $\pi_3(SU(N)) = \mathbb{Z}$  (see e.g. [51, 52]). Use of the expression Eq. (2.1) in the  $G_2$  gauge theory would however give only even topological charges. The correct normalization to be used in the general case has been discussed in [53] and the final result is

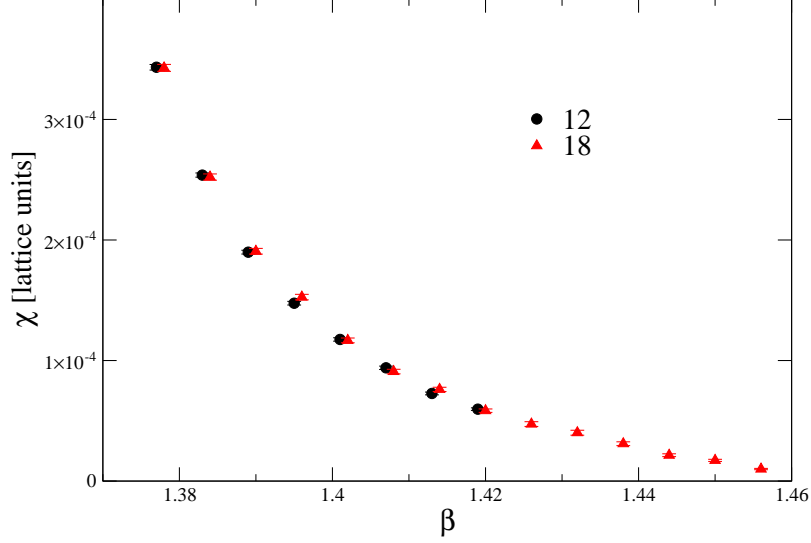
$$q(x) = \frac{g^2}{64K\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left[ F_{\mu\nu}(x) F_{\rho\sigma}(x) \right] , \quad (3.5)$$

where the algebra generators  $T^a$ s are normalized in such a way that the longest root is equal to 1 and  $K$  is given by the relation  $\text{Tr}(T^a T^b) = K\delta^{ab}$ . Using the explicit realization given in [46] of the  $G_2$  algebra it is simple to show that for  $G_2$  we have  $K = 1$  (while  $K = 1/2$  for  $SU(N)$ ).

On the lattice several methods exist to associate a value of the topological charge to a given configuration. Since all these methods has been proven to give equivalent results (see e.g [24, 54]) and high statistics is needed in our study, we adopted the cheapest from the numerical point of view. The topological charge has been measured after cooling and the simplest discretization with definite parity of Eq. (3.5) was adopted, namely [55]

$$q_L(x) = -\frac{1}{2^4 \times 64K\pi^2} \sum_{\mu\nu\rho\sigma=\pm 1}^{\pm 4} \tilde{\epsilon}_{\mu\nu\rho\sigma} \text{Tr} (U_{\mu\nu}(x) U_{\rho\sigma}(x)) , \quad (3.6)$$

where  $\tilde{\epsilon}_{\mu\nu\rho\sigma}$  coincides with the usual Levi-Civita tensor for positive indices, while for the negative directions it is defined by the relation  $\tilde{\epsilon}_{\mu\nu\rho\sigma} = -\tilde{\epsilon}_{(-\mu)\nu\rho\sigma}$  and the complete anti-symmetry. This approach is the same adopted e.g. in [54], to which we refer for technical



**Figure 1.** Zero temperature topological susceptibility measured on  $12^4$  and  $18^4$  lattices. Data measured on the  $18^4$  lattice have been slightly shifted to the right for clarity.

details, the only difference being that, for the  $G_2$  gauge theory, cooling consists of cooling on three  $SU(2) \subset G_2$  and random gauge transformations.

## 4 Numerical results

In this section we will present the numerical results obtained by means of Monte-Carlo simulations of the  $G_2$  lattice theory. Most of the finite temperature simulations have been performed by using lattices with temporal extent<sup>3</sup>  $N_t = 6$ ; by using three different values for the spatial size,  $N_s = 12, 18, 24$ , we verified that an aspect ratio  $N_s/N_t = 3$  is sufficient to neglect finite size effects with our statistical uncertainties. To check for the continuum limit a  $8 \times 24^3$  lattice was then simulated.

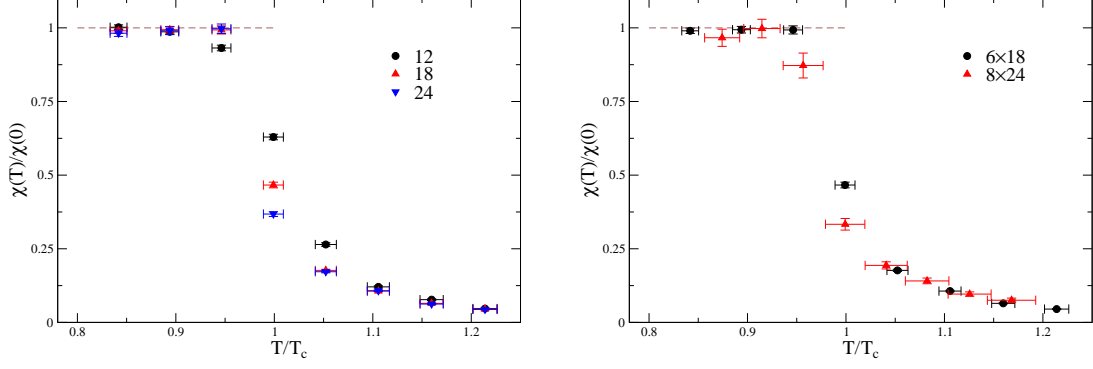
The results reported in the following have been obtained by using the topological charge extracted after 90 cooling steps (see [54] for more details on the procedure adopted), but they proved to be stable within errors for a number of cooling steps in the range from 30 to 150. A statistics of  $\mathcal{O}(10^5)$  measures have been used for each coupling value, with measures performed every 10 update steps and each step consisting of a Cabibbo-Marinari heatbath, five Cabibbo-Marinari overrelaxations and a random gauge transformation.

The values of  $T/T_c$  have been estimated by using the parametrization reported in [22] for the string tension and the following values of the critical couplings:  $\beta_c(N_t = 6) = 1.3951(2)$  (see [16]) and  $\beta_c(N_t = 8) = 1.431(3)$  (compatible with the one reported in [22]).

The first observable studied has been the topological susceptibility  $\chi(T)$  and, in particular, the dimensionless ratio  $\chi(T)/\chi(T = 0)$ . In order to compute this ratio simulations have been performed on symmetric lattices at the same coupling values adopted for the

<sup>3</sup>This is the smallest  $N_t$  value for which the finite temperature transition takes place at a critical coupling larger than the one corresponding to the bulk transition, see e.g. [16].



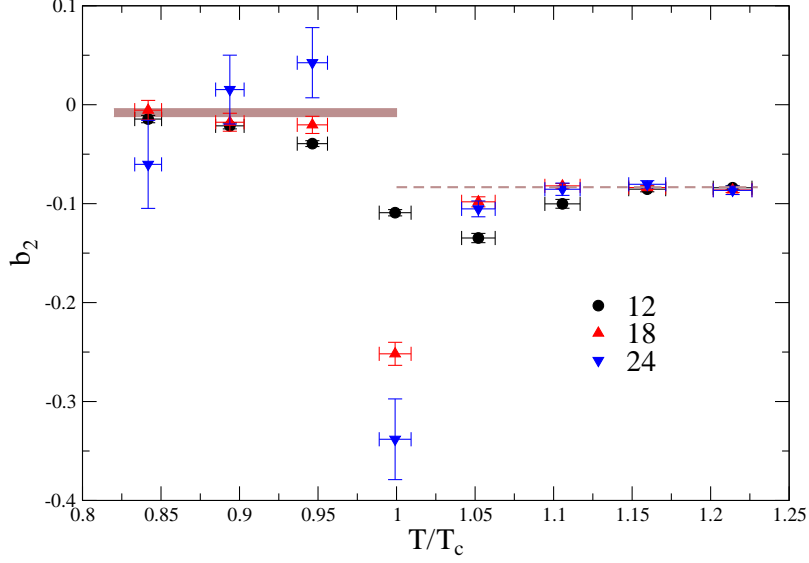


**Figure 2.** Ratio of the finite temperature topological susceptibility and the zero temperature one. (*Left*) Results for the lattices  $6 \times N_s^3$ , with  $N_s = 12, 18, 24$ . (*Right*) Comparison of the results obtained on  $6 \times 18^3$  and  $8 \times 24^3$  lattices.

finite temperature runs. The results obtained for  $\chi(T = 0)$  on the lattices  $12^4$  and  $18^4$  are shown in Fig. (1). From the comparison of the results obtained on the two lattices we can conclude that the  $12^4$  data do not show significant finite size effects. Moreover the size of the  $18^4$  lattice at  $\beta = 1.455$  ( $\approx 1.5$  fm) is larger than the one of the  $12^4$  lattice at  $\beta = 1.419$  ( $\approx 1.3$  fm); we thus conclude that also the  $18^4$  data are not affected by significant finite size effects.

In Fig. (2) the estimated values of the ratio  $\chi(T)/\chi(0)$  are shown. The left panel displays the results obtained on the  $N_t = 6$  lattices for several spatial extents and, as previously anticipated, no significant finite size effects can be seen as far as the aspect ratio is 3 or larger (apart from the data at  $T \approx T_c$ ). In the right panel of Fig. (2) we compare the results obtained by using two different lattice spacing at fixed physical lattice size. The nice agreement between the determinations obtained by using the  $6 \times 18^3$  and the  $8 \times 24^3$  lattices supports the absence of significant lattice artefacts in our measurements. We can thus conclude that, like in  $SU(N)$  gauge theories,  $\chi(T)$  stays constant for  $T < T_c$ , with an abrupt decrease at deconfinement.

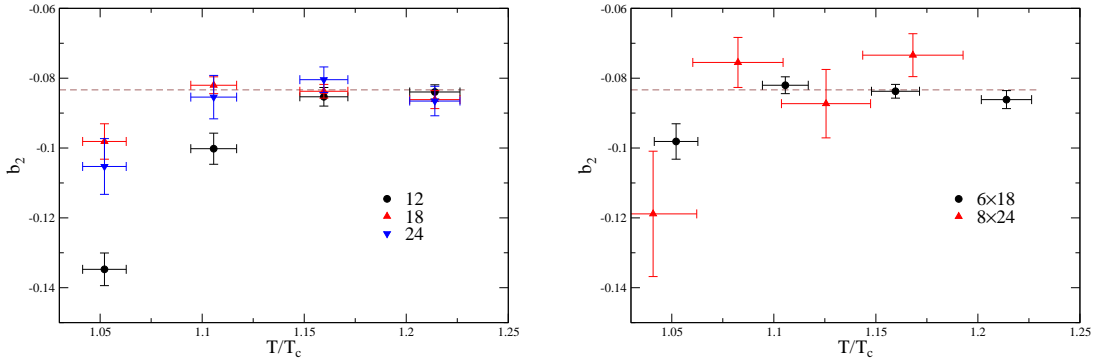
The value of the parameter  $b_2$  across the deconfinement transition is shown in Fig. (3) for the  $N_t = 6$  lattices. As a first observation we notice that, although the statistics (and the autocorrelations) are of the same order of magnitude for all the lattice sizes studied, the error bars of the low temperature data for the  $6 \times 24^3$  lattice are much larger than the ones for the smaller lattices. This phenomenon is known to happen also in  $SU(N)$  LGTs and is likely related to the peculiar form Eq. (2.4) of the  $b_{2n}$  observables when computed by means of simulations at  $\theta = 0$ . A possible way out could be to perform simulations at imaginary value of the  $\theta$  parameter, as proposed in [39], however it would be difficult to study in this way the neighbourhood of the deconfinement transition, since the value of the deconfinement temperature  $T_c$  also depend on  $\theta$  (see [56, 57] for lattice studies and [58–60] for other approaches). Since for all the more standard observables the results obtained on the lattice  $6 \times 18^3$  are completely equivalent (apart from the  $T \approx T_c$  data) to the ones obtained on the larger  $6 \times 24^3$  lattice, we expect that, also for  $b_2$ , data obtained on the lattice with aspect ratio 3 do not suffer from severe finite size effects and we will refer to



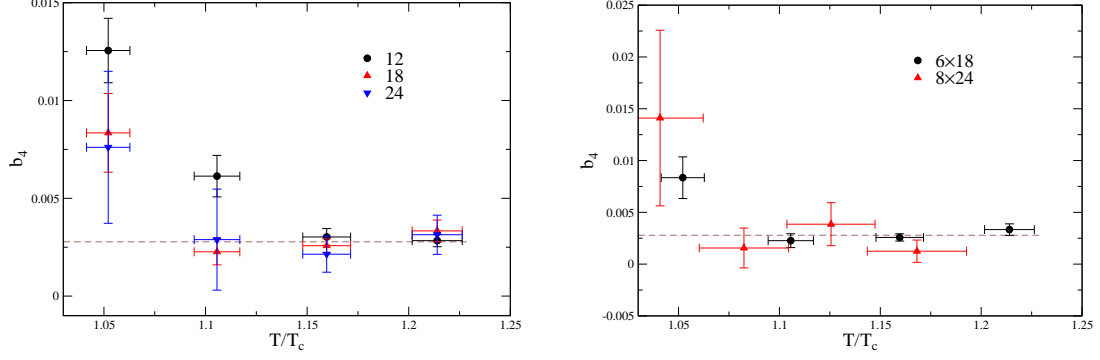
**Figure 3.** Plot of the  $b_2$  value across the deconfinement temperature for lattices  $6 \times N_s^3$ , with  $N_s = 12, 18, 24$ . The band for  $T/T_c \leq 1$  is the  $b_2$  value for  $SU(6)$  (see [40]), while the dashed line for  $T/T_c > 1$  denotes the dilute instanton gas value  $b_2^{(\text{dig})} = -1/12$ .

them in the following discussions.

Let us now come to the behaviour of  $b_2$  across the transition. In the low temperature phase the value of  $b_2$  is quite small, almost compatible with zero, much like what happens in  $SU(N)$  gauge theories; for reference the value of  $b_2$  for  $SU(6)$  at zero temperature ( $b_2 = 0.008(4)$ , [40]) is also shown in Fig. (3). For  $T > T_c$  the value of  $b_2$  increases by an order of magnitude and promptly approaches the asymptotic value predicted by the dilute instanton gas,  $b_2^{(\text{dig})} = -1/12$ . To better appreciate the rapidity of the convergence to the asymptotic value, in Fig. (4) we display a magnification of the high temperature region, both for  $N_t = 6$  and  $N_t = 8$  lattices. Significant deviations from the value  $b_2^{(\text{dig})}$  are visible only for  $T \lesssim 1.1T_c$ .



**Figure 4.**  $b_2$  value in the high temperature region. The horizontal dashed line denotes the dilute instanton gas result  $b_2^{(\text{dig})} = -1/12$ . (Left) Results for the lattices  $6 \times N_s^3$ , with  $N_s = 12, 18, 24$ . (Right) Comparison of the results obtained on  $6 \times 18^3$  and  $8 \times 24^3$  lattices.



**Figure 5.**  $b_4$  value in the high temperature region. The horizontal dashed line denotes the dilute instanton gas result  $b_4^{(\text{dig})} = 1/360$ . (Left) Results for the lattices  $6 \times N_s^3$ , with  $N_s = 12, 18, 24$ . (Right) Comparison of the results obtained on  $6 \times 18^3$  and  $8 \times 24^3$  lattices.

The numerical estimates of  $b_4$  in the low temperature phase are too noisy to extract from them useful informations, but high temperature data are precise enough to show the convergence to the dilute instanton gas value  $b_4^{(\text{dig})} = 1/360$ , see Fig. (5). Like for the case of  $b_2$ , no significant discrepancies from the asymptotic value  $b_4^{(\text{dig})}$  can be seen as far as  $T \gtrsim 1.1T_c$ .

## 5 Conclusions

In this work we studied the behaviour, across the deconfinement transition, of the topological susceptibility  $\chi(T)$  and of the coefficients  $b_{2n}(T)$  that parametrize the  $\theta$  dependence of the free energy in the  $G_2$  gauge theory.

The sudden drop at deconfinement of the ratio  $\chi(T)/\chi(T=0)$  signals that in the high temperature phase the topological activity is strongly suppressed. The abrupt change of the  $b_2$  coefficient at  $T_c$  shows that the difference between the low and the high temperature phases is not just a difference in the global activity, but that also the functional form of the  $\theta$  dependence of the free energy has changed. The  $b_2$  and  $b_4$  values in the deconfined phase rapidly approach the predictions of the dilute instanton gas model, which well reproduce lattice data for  $T \gtrsim 1.1T_c$ .

The picture that emerges is surprisingly similar to that of the  $SU(N)$  gauge theory, with however a fundamental difference: in the  $SU(N)$  setup the change of the  $\theta$  dependence at deconfinement can be conveniently interpreted as a change from a low temperature large- $N$  regime to an high temperature instanton one. While the argument for the instanton-like behaviour of the free energy in the deconfined phase can be applied without modifications to the  $G_2$  gauge theory, it is not clear what takes the role of the large- $N$  regime for this theory.

The most natural explanation of this common behaviour of  $SU(N)$  and  $G_2$  theories is probably that the confinement mechanism is the same for all the simple gauge groups and that the degrees of freedom responsible for the confinement have non-trivial topological properties. Several proposals that go in this directions exist in the literature and are

$\beta$	$T/T_c$	$\chi(T)/\chi(0)$	$b_2$	$b_4$
1.377	0.8418(85)	1.0021(80)	-0.0145(37)	-0.0022(23)
1.383	0.8938(91)	0.9866(83)	-0.0214(31)	-0.0007(16)
1.389	0.9463(96)	0.9317(95)	-0.0393(30)	-0.0006(18)
1.395	0.999(10)	0.6294(93)	-0.1092(32)	0.0008(11)
1.401	1.052(11)	0.2644(60)	-0.1347(47)	0.0126(16)
1.407	1.106(11)	0.1206(35)	-0.1002(44)	0.0061(11)
1.413	1.160(12)	0.0773(28)	-0.0853(27)	0.00302(43)
1.419	1.214(12)	0.0469(20)	-0.0839(20)	0.00284(31)

**Table 1.** Data for the lattice  $6 \times 12^3$ .

actively investigated, like dyons [8], bions [9, 10, 58], instanton-quarks [42, 61] and the relations between monopoles and instantons [62], not to mention the analogy between the  $\theta$  angle and the chemical potential noted in [57] and the similarities with spin models [63–66].

A related point is the global analytical structure of the free energy as a function of  $\theta$  in the confined phase. We previously recalled the large- $N$  Witten’s argument, which suggests that the free energy of the  $SU(N)$  theory is a multi-branched function of the form Eq. (2.11) and gives a qualitative explanation of the smallness of  $b_2$  for  $T < T_c$ . It is tempting to relate the small value of  $b_2$  observed also in the low temperature phase of the  $G_2$  theory to an analogous multi-branched structure of the free energy, which however have no natural large- $N$  interpretation.

The early onset of the dilute instanton gas regime, just slightly above the deconfinement transition, also appears to be a feature common to both  $SU(N)$  and  $G_2$  gauge theories, with indications that this is true also in presence of quarks [67, 68]. While it is not clear if these two common features, i.e. the change of  $\theta$  dependence at deconfinement and the early onset of the instanton behaviour, are related to each other or not, it is interesting to notice that also deviations from the dilute instanton gas behaviour are qualitatively similar in  $SU(N)$  and  $G_2$  gauge theories, with  $b_2$  approaching its asymptotic value from below and  $b_4$  from above.

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## A Numerical data

In Tabs. (1),(2),(3) and (4) the values of the bare coupling used for the various lattices and the estimates obtained for  $T/T_c$ ,  $\chi(T)/\chi(T=0)$ ,  $b_2$  and  $b_4$  are reported.

$\beta$	$T/T_c$	$\chi(T)/\chi(0)$	$b_2$	$b_4$
1.377	0.8418(85)	0.9901(91)	-0.0055(98)	-0.034(18)
1.383	0.8938(91)	0.994(11)	-0.0177(90)	0.019(15)
1.389	0.9463(96)	0.993(14)	-0.0204(86)	0.0059(97)
1.395	0.999(10)	0.4664(97)	-0.252(12)	0.045(14)
1.401	1.052(11)	0.1767(36)	-0.0981(51)	0.0083(20)
1.407	1.106(11)	0.1065(25)	-0.0820(24)	0.00227(67)
1.413	1.160(12)	0.0647(18)	-0.0837(20)	0.00258(35)
1.419	1.214(12)	0.0452(15)	-0.0861(26)	0.00333(56)

**Table 2.** Data for the lattice  $6 \times 18^3$ .

$\beta$	$T/T_c$	$\chi(T)/\chi(0)$	$b_2$	$b_4$
1.377	0.8418(85)	0.982(11)	-0.060(45)	-0.29(20)
1.383	0.8938(91)	0.990(13)	0.015(35)	-0.15(11)
1.389	0.9463(96)	0.997(16)	0.042(35)	0.132(87)
1.395	0.999(10)	0.368(81)	-0.338(41)	0.270(88)
1.401	1.052(11)	0.1721(38)	-0.1053(80)	0.0076(39)
1.407	1.106(11)	0.1074(28)	-0.0854(62)	0.0029(26)
1.413	1.160(12)	0.0623(19)	-0.0804(37)	0.00214(92)
1.419	1.214(12)	0.0442(16)	-0.0865(42)	0.0031(10)

**Table 3.** Data for the lattice  $6 \times 24^3$ .

$\beta$	$T/T_c$	$\chi(T)/\chi(0)$	$b_2$	$b_4$
1.413	0.873(18)	0.966(30)	-0.114(45)	0.065(72)
1.419	0.916(18)	0.997(31)	0.022(23)	-0.025(20)
1.425	0.956(20)	0.872(42)	-0.053(22)	-0.034(15)
1.431	0.997(20)	0.333(20)	-0.097(13)	-0.0041(35)
1.437	1.040(21)	0.194(12)	-0.119(18)	0.0141(85)
1.443	1.082(22)	0.1409(96)	-0.0755(71)	0.0016(19)
1.449	1.125(22)	0.0962(73)	-0.0873(98)	0.0039(21)
1.455	1.168(25)	0.0753(72)	-0.0734(41)	0.0012(11)

**Table 4.** Data for the lattice  $8 \times 24^3$ .

## References

- [1] L. D. McLerran and B. Svetitsky, Phys. Rev. D **24**, 450 (1981).
- [2] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
- [3] J. Greensite, Prog. Part. Nucl. Phys. **51**, 1 (2003) [hep-lat/0301023].
- [4] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B **210**, 423 (1982).
- [5] P. de Forcrand and O. Jahn, Nucl. Phys. B **651**, 125 (2003) [hep-lat/0211004].

- [6] A. Di Giacomo, L. Lepori and F. Pucci, JHEP **0810**, 096 (2008) [arXiv:0810.4226 [hep-lat]].
- [7] C. Bonati, A. Di Giacomo, L. Lepori and F. Pucci, Phys. Rev. D **81**, 085022 (2010) [arXiv:1002.3874 [hep-lat]].
- [8] D. Diakonov and V. Petrov, AIP Conf. Proc. **1343**, 69 (2011) [arXiv:1011.5636 [hep-th]].
- [9] E. Poppitz, T. Schäfer and M. Ünsal, JHEP **1303**, 087 (2013) [arXiv:1212.1238].
- [10] M. M. Anber, E. Poppitz and B. Teeple, JHEP **1409**, 040 (2014) [arXiv:1406.1199 [hep-th]].
- [11] K. Holland, P. Minkowski, M. Pepe and U.-J. Wiese, Nucl. Phys. B **668**, 207 (2003) [hep-lat/0302023].
- [12] M. Pepe, PoS LAT **2005**, 017 (2006) [Nucl. Phys. Proc. Suppl. **153**, 207 (2006)] [hep-lat/0510013].
- [13] J. Greensite, K. Langfeld, S. Olejnik, H. Reinhardt and T. Tok, Phys. Rev. D **75**, 034501 (2007) [hep-lat/0609050].
- [14] M. Pepe and U.-J. Wiese, Nucl. Phys. B **768**, 21 (2007) [hep-lat/0610076].
- [15] G. Cossu, M. D’Elia, A. Di Giacomo, C. Pica and B. Lucini, PoS LAT **2006**, 063 (2006) [hep-lat/0609061].
- [16] G. Cossu, M. D’Elia, A. Di Giacomo, B. Lucini and C. Pica, JHEP **0710**, 100 (2007) [arXiv:0709.0669 [hep-lat]].
- [17] A. Maas and S. Olejnik, JHEP **0802**, 070 (2008) [arXiv:0711.1451 [hep-lat]].
- [18] L. Liptak and S. Olejnik, Phys. Rev. D **78**, 074501 (2008) [arXiv:0807.1390 [hep-lat]].
- [19] J. Danzer, C. Gattringer and A. Maas, JHEP **0901**, 024 (2009) [arXiv:0810.3973 [hep-lat]].
- [20] B. H. Wellegehausen, A. Wipf and C. Wozar, Phys. Rev. D **83**, 016001 (2011) [arXiv:1006.2305 [hep-lat]].
- [21] E. M. Ilgenfritz and A. Maas, Phys. Rev. D **86**, 114508 (2012) [arXiv:1210.5963 [hep-lat]].
- [22] M. Bruno, M. Caselle, M. Panero and R. Pellegrini, arXiv:1409.8305 [hep-lat].
- [23] C. Vafa and E. Witten, Phys. Rev. Lett. **53**, 535 (1984).
- [24] E. Vicari and H. Panagopoulos, Phys. Rept. **470**, 93 (2009) [arXiv:0803.1593 [hep-th]].
- [25] D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
- [26] B. Alles, M. D’Elia and A. Di Giacomo, Nucl. Phys. B **494**, 281 (1997) [Erratum-ibid. B **679**, 397 (2004)] [hep-lat/9605013].
- [27] B. Alles, M. D’Elia and A. Di Giacomo, Phys. Lett. B **412**, 119 (1997) [hep-lat/9706016].
- [28] B. Alles, M. D’Elia and A. Di Giacomo, Phys. Lett. B **483**, 139 (2000) [hep-lat/0004020].
- [29] C. Gattringer, R. Hoffmann and S. Schaefer, Phys. Lett. B **535**, 358 (2002) [hep-lat/0203013].
- [30] L. Del Debbio, H. Panagopoulos and E. Vicari, JHEP **0409**, 028 (2004) [hep-th/0407068].
- [31] B. Lucini, M. Teper and U. Wenger, Nucl. Phys. B **715**, 461 (2005) [hep-lat/0401028].
- [32] G. ’t Hooft, Nucl. Phys. B **72**, 461 (1974).
- [33] B. Lucini and M. Panero, Phys. Rept. **526**, 93 (2013) [arXiv:1210.4997 [hep-th]].
- [34] E. Witten, Annals Phys. **128**, 363 (1980).

- [35] E. Witten, Phys. Rev. Lett. **81**, 2862 (1998) [hep-th/9807109].
- [36] L. Del Debbio, H. Panagopoulos and E. Vicari, JHEP **0208**, 044 (2002) [hep-th/0204125].
- [37] M. D’Elia, Nucl. Phys. B **661**, 139 (2003) [hep-lat/0302007].
- [38] L. Giusti, S. Petrarca and B. Taglienti, Phys. Rev. D **76**, 094510 (2007) [arXiv:0705.2352 [hep-th]].
- [39] H. Panagopoulos and E. Vicari, JHEP **1111**, 119 (2011) [arXiv:1109.6815 [hep-lat]].
- [40] C. Bonati, M. D’Elia, H. Panagopoulos and E. Vicari, Phys. Rev. Lett. **110**, 252003 (2013) [arXiv:1301.7640 [hep-lat]].
- [41] O. Bergman and G. Lifschytz, JHEP **0704**, 043 (2007) [hep-th/0612289].
- [42] A. Parnachev and A. R. Zhitnitsky, Phys. Rev. D **78**, 125002 (2008) [arXiv:0806.1736 [hep-ph]].
- [43] E. Thomas and A. R. Zhitnitsky, Phys. Rev. D **85**, 044039 (2012) [arXiv:1109.2608 [hep-th]].
- [44] J. F. Adams “Lectures on exceptional Lie groups” The University of Chicago Press (1996).
- [45] F. Reese Harvey “Spinors and Calibrations” Academic Press (1990).
- [46] S. L. Cacciatori, B. L. Cerchiai, A. Della Vedova, G. Ortenzi and A. Scotti, J. Math. Phys. **46**, 083512 (2005) [hep-th/0503106].
- [47] N. Cabibbo and E. Marinari, Phys. Lett. B **119**, 387 (1982).
- [48] M. Creutz, Phys. Rev. D **21**, 2308 (1980).
- [49] A. D. Kennedy and B. J. Pendleton, Phys. Lett. B **156**, 393 (1985).
- [50] M. Creutz, Phys. Rev. D **36**, 515 (1987).
- [51] E. J. Weinberg “Classical solutions in quantum field theory” Cambridge University Press (2012).
- [52] S. Weinberg “The quantum theory of fields, volume II. Modern Applications” Cambridge University Press (1969)
- [53] C. W. Bernard, N. H. Christ, A. H. Guth and E. J. Weinberg, Phys. Rev. D **16**, 2967 (1977).
- [54] C. Bonati and M. D’Elia, Phys. Rev. D **89**, 105005 (2014) [arXiv:1401.2441 [hep-lat]].
- [55] P. Di Vecchia, K. Fabricius, G. C. Rossi and G. Veneziano, Nucl. Phys. B **192**, 392 (1981).
- [56] M. D’Elia and F. Negro, Phys. Rev. Lett. **109**, 072001 (2012) [arXiv:1205.0538 [hep-lat]].
- [57] M. D’Elia and F. Negro, Phys. Rev. D **88**, 034503 (2013) [arXiv:1306.2919 [hep-lat]].
- [58] M. Ünsal, Phys. Rev. D **86**, 105012 (2012) [arXiv:1201.6426 [hep-th]].
- [59] M. M. Anber, Phys. Rev. D **88**, 085003 (2013) [arXiv:1302.2641 [hep-th]].
- [60] F. Bigazzi and A. L. Cotrone, JHEP **1501**, 104 (2015) [arXiv:1410.2443 [hep-th]].
- [61] A. R. Zhitnitsky, Nucl. Phys. A **813**, 279 (2008) [arXiv:0808.1447 [hep-ph]].
- [62] A. Di Giacomo and M. Hasegawa, arXiv:1501.06517 [hep-lat].
- [63] F. Bruckmann, Phys. Rev. Lett. **100**, 051602 (2008) [arXiv:0707.0775 [hep-th]].
- [64] G. V. Dunne and M. Ünsal, JHEP **1211**, 170 (2012) [arXiv:1210.2423 [hep-th]].

- [65] A. Cherman, D. Dorigoni, G. V. Dunne and M. Ünsal, Phys. Rev. Lett. **112**, 021601 (2014) [arXiv:1308.0127 [hep-th]].
- [66] F. Bruckmann and T. Sulejmanpasic, Phys. Rev. D **90**, 105010 (2014) [arXiv:1408.2229 [hep-th]].
- [67] G. Cossu, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, H. Matsufuru and J. I. Noaki, Phys. Rev. D **87**, 114514 (2013) [arXiv:1304.6145 [hep-lat]].
- [68] T. Kanazawa and N. Yamamoto, arXiv:1410.3614 [hep-th].